

When Do the CR Equations Imply Analyticity?

The Multivariable Differentiability Theorem, Looman–Menchoff, and Consequences

Lecture Notes

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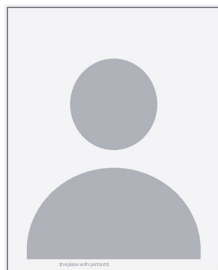
1 A Real Analysis Refresher (continued)

Theorem (Multivariable Calculus)

Let $u(x, y)$ have partial derivatives $\frac{\partial u}{\partial x}(x, y)$ and $\frac{\partial u}{\partial y}(x, y)$ for all $(x, y) \in B((x_0, y_0), \delta)$ which are **continuous** at (x_0, y_0) . Then u is differentiable at (x_0, y_0) .

Working criterion. Combining this with the Cauchy–Riemann theorem: if the partials of u and v exist and are continuous on $B(z_0, r)$ and satisfy the Cauchy–Riemann equations there, then $f = u + iv$ is analytic in $B(z_0, r)$.

2 Do the CR Equations Alone Imply Analyticity?



Herman Looman
(fl. 1923)



Dmitrii Menshov
(1892–1988)

The working criterion above assumes real-differentiability (secured by continuous partials). Can we drop it? Not naively.

Remark — **some regularity must be assumed a priori.** There exists f whose Wirtinger derivatives $\frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar{z}}$ exist everywhere, with $\frac{\partial f}{\partial \bar{z}} \equiv 0$, yet f is *not* everywhere analytic. Looman’s example is

$$f(z) = \begin{cases} e^{-1/z^4}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

(This is the same “flat function” trick behind the standard real-analysis example e^{-1/x^2} , smooth but not real-analytic at 0.) Away from the origin f is a composition of analytic functions, so $\partial f / \partial \bar{z} = 0$ there. At the origin the partials exist and vanish (along both axes $z^4 > 0$, so $f \rightarrow 0$ smoothly), hence $\partial f / \partial \bar{z}(0) = 0$

as well. Nevertheless f is not even continuous at 0: along the ray $\arg z = \pi/4$ we have $z^4 = -r^4$, so $|f(z)| = e^{1/r^4} \rightarrow \infty$. The point is that the partials fail to be *continuous* at 0 — so the working criterion does not apply, and there is no contradiction.

Continuity of f itself, however, *is* enough — a deep and surprising fact.

Theorem (Looman–Menchoff)

Let $f = u + iv$ be **continuous** on $B(z_0, r)$, suppose all the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ **exist** on $B(z_0, r)$, and suppose the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold there. Then f is analytic in $B(z_0, r)$.

No continuity of the partials, and no a priori differentiability, is required — only continuity of f , existence of the partials, and the Cauchy–Riemann equations. (Due to H. Looman, 1923, and D. Menshov, 1936; it strengthens an earlier theorem of Goursat.)

3 Vanishing Derivative and Constants

Theorem

Let f be analytic in $B(z_0, \delta)$ with $f'(z) \equiv 0$ for all $z \in B(z_0, \delta)$. Then $f(z) \equiv \text{const.}$

Proof. From $f'(z) = 0$,

$$\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x} \equiv 0 \implies \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} \equiv 0.$$

By calculus, a function on a (connected) disc with vanishing gradient is constant, so $u \equiv \text{const}$ and $v \equiv \text{const}$. ■

Remark — one can assume less. It is enough to require

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \equiv 0$$

without assuming differentiability a priori: the differentiability follows from continuity.

Proof. Since $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ vanish identically they are continuous, so by the calculus theorem f is real-differentiable. Then

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) = 0 \implies f \text{ is analytic,}$$

while $\frac{\partial f}{\partial z} = \frac{1}{2}(\partial_x - i \partial_y)f = 0 = f'$, hence f is constant. ■

4 Examples

- **The exponential is entire.** For $f(z) = e^x(\cos y + i \sin y)$ we have $u = e^x \cos y$, $v = e^x \sin y$, with continuous partials satisfying

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$

Hence e^z is entire, and $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = e^z$.

- **Conjugation is nowhere differentiable.** For $f(z) = \bar{z} = x - iy$ we get $u_x = 1$ but $v_y = -1$, so the first CR equation fails everywhere; equivalently $\partial f / \partial \bar{z} = 1 \neq 0$. Thus $\bar{z}$ is differentiable nowhere.
- **Differentiable at one point only.** For $f(z) = |z|^2 = z\bar{z}$ we have $u = x^2 + y^2$, $v = 0$. The CR equations read $2x = 0$ and $2y = 0$, satisfied only at the origin. So f is complex differentiable at $z = 0$ (with $f'(0) = 0$) and nowhere else — analytic at no point of the plane. Differentiability at a point does not propagate to a neighborhood.